Modeling Human Behavior in Economics and Social Science

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Abstract

The complex interactions between human behaviors and social economic sciences is critically analyzed in this paper in view of possible applications of mathematical modeling as an attainable interdisciplinary approach to understand and simulate the aforementioned dynamics. The quest is developed along three steps: Firstly an overall analysis of social and economic sciences indicates the main requirements that a contribution of mathematical modeling should bring to these sciences; subsequently the focus moves to an overview of mathematical tools and to the selection of those which appear, according to the authors bias, appropriate to the modeling; finally, a survey of applications is presented looking ahead to research perspectives.

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JEL Classification: C02, C63, C68

1. Plan of the Paper

This paper presents a critical analysis and a survey of the research activity and perspectives coming from the actual and potential interaction between hard sciences, such as mathematics and physics, and social sciences, such as economics, politics and more generally all sciences where human behavior plays the important role of determining complex emerging dynamics. Our proposal is motivated by recent radical changes in the modeling of social and economical sciences, leading to interesting cultural developments in the social sciences. This new trend suggests, and from the researcher's viewpoint somehow imposes, the necessity to include the complexity of behavioral features of these systems in the research activity, see [6, 96, 129, 137]. A wide interesting literature stems from this radical change, and a variety of interesting dynamics have been explored, including the role of ethical and unethical behaviors [80, 115, 122, 125], or the interplay of alternative social dynamics [74, 76, 109]. Moreover, various authors have remarked that the interactions across individuals, or entities in general, are not simply limited to binary communications; in fact individual entities are also sensitive and reactive to the behavior

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of the systems where the interaction takes place [9, 75, 99] and the reaction is also enhanced by the action of the networks [2, 3, 11, 26, 75, 134].

Social sciences are classified as *soft sciences,* in contrast with the so called *hard sciences* [73]; our paper proposes a working environment generalizing the approach. We follow the initial suggestion of [9] by taking the scientific environment under consideration as that of *sciences of living systems*. This definition goes beyond the concept of strict, and somewhat schematic, dichotomy of sciences as soft and hard [65, 64]. Moreover, [9] focuses on the mathematical approach to the modeling of social sciences and proposes a new systems approach; this is a crucial reference to our paper, that refers to a broader context, where economics, politics, and other social sciences operate. More in detail, our paper looks at these sciences within the framework of living systems that are, by intrinsic nature, complex. We hence agree with the rationale of [9], namely that a constructive interplay between mathematics and the socio-economics analysis can be properly developed as far as mathematical tools help to take into account the complexity features of behavioral systems. A successful interaction can lead to exploratory models which can depict qualitatively the behavior of social-economical systems for by exploring their asymptotic behaviors corresponding to different choice of the parameters.

According to [9], exploratory models exploit the trends of certain dynamics focusing on qualitative behaviors as function of time, where an exploratory investigation can study the effects of different decisions. Additional analysis is required by the design of predictive models that need a detailed quantitative assessment of parameters to describe quantitatively the behavior of systems. It is known that the classical deterministic mathematical tools, based on causality principles, generally fail when dealing with the aforementioned objectives. Therefore, additional work is needed to chase the perspective ideas posed in [9]. In the present survey, a critical overview of the existing literature is proposed, with the aim at proposing new research perspectives. This short introduction allows us to define more precisely the content of our paper, that it is structured as follows:

Section 2 provides a description of some basic features of social and, in particular, economic systems related to the general framework of the theory of living, hence complex, systems. More in detail, we give a description of the main complex features of heterogeneous behaviors that the modeling approach should account for, as opposed to the simplifying hypotheses that are usually adopted at the outset to make the question under analysis easier to model from the mathematical point of view. The following topics are treated: rationality vs. bounded rationality, homogeneity vs. heterogeneity, equilibrium vs. out-of-equilibrium and linearity vs. non-linearity. Finally, we briefly look at the role of computation. The presentation does not propose a formalization, which is left for the next sections, although in specific contexts.

Section 3 presents a critical overview of a variety of mathematical tools. Namely, population dynamics, population dynamics with internal structure, game theory, evolutionary games, mean field games, statistical dynamics, kinetic theory. This overview also discusses pros and cons of the mathematical tools and explain why the kinetic theory approach has been chosen for the modeling approach. This section presents the rationale of different quantitative tools.

Section 4 is provides a technical presentation of the tools of the kinetic theory of active particles. The contents are proposed at a formalized level referring to the existing literature. Moreover, the rationale of the mathematical methods is explained and critically analyzed so that this section is addressed to a broad audience and not simply to devoted mathematicians. This section aims at providing the conceptual mathematical framework for the derivation of models.

Section 5 presents and discusses developments and applications of the mathematical tools presented in Sec. 3 showing how the modeling approach has been applied in recent years to the study of a variety of social and political systems. This overview shows how applications to economics are still confined at a primitive stage, still waiting for possible development. This critical survey accounts for the role of human behaviors on the overall dynamics and aims at showing how these "social behaviors" and "political choices" can have an important influence on the dynamics of our society. The analysis of two case studies contributes to develop the critical analysis in view of the research perspectives presented in the last section.

Section 6 looks ahead to research perspectives that are generated by the interaction between hard and soft sciences. Various topics are brought to the attention of the reader. The most interesting is, we argue, the role of networks in economics, finance and, in general, social sciences.

2. Main issues toward mathematical modeling

This section discusses some issues related to the modeling of socio-economic systems. These are chosen among the many possible, because in our view they represent open and challenging questions. The state of the art of the scientific literature suggests that further development toward a general mathematical structure is needed to deal with a satisfactory comprehension of the former. This structure is deemed to capture the complexity features of living systems in general and of socio-economical systems in particular.

Beyond the representative agent: the role of heterogeneity. The approach behind the analysis of the behavior of a *Representative Agent* is that each individual in the social and economic system behaves as the average individual, that in turn decides the most rational strategy given the set of public available information. The *Walrasian general equilibrium* is a prominent example of a model based on the assumption of a representative agent, as well as the *Neoclassical growth model*. On the other hand variables such as information, ability and income are known to be heterogeneously distributed across individuals in social systems. Heterogeneity plays an important role in the collective dynamics [9, 97]. The reduction of a group of heterogeneous agents to the representative individual is widely debated across the scientific literature (see [96, 83] and references therein). Alternative statistical tools introduce different levels of aggregation which help accounting for heterogeneity. The first one is obtained by clustering the agents according to some observable metrics; different subsystems are then put in relationship with each other (see [99] and see [75, 76] for examples in political economy and opinion formation). The role that heterogeneity can play is also taken into account in the evolutionary dynamics literature [123]. In this case heterogeneity is often associated to the idea of some individuals interacting more than others in the population so that the population is heterogeneously clustered with respect to the interactions. Exploring the impact of heterogeneity in empirical applications is important to identify the type of heterogeneity which is relevant for alternative fields of social sciences. [50] explores, for instance, heterogeneity in individual utility, income, wealth, and preference for risk, as well as, in the individuals' market participation, that in turn correlates to the analysis of consumption, saving, and labor market participation.

Beyond the homo oeconomicus: the role of bounded rationality. Homo oeconomicus may be taken as the individual who behaves rationally based upon the set of available information [90], for example, s/he maximizes her/his utility function given the set of constraints, the set of information, and the distribution of probability of future outcomes. In this sense, rationality correlates to the approach of expected utility [136]. This view is challenged since [95] who suggests that, if the analysis of behavior of human beings in financial markets is of interest, then cognitive processes must be taken into adequate account. From this idea, *behavioral finance* theories has been widely developed. Kanheman highlighted the role of intuitive judgment. His research, based on experimental observations, did show that the the day by day attitude toward risk of the individual systematically violates the assumptions of the *expected utility model*. The basic definition of risk aversion, as given by the concavity of the utility function, has been put at question (for a more general discussion on the role of cognitive processes in behavioral sciences see [88]). *Homo oeconomicus* is shown to struggle with, among others, deterministic chaos, incomplete information, finite memory, limited processing capabilities; emotions affecting the output of decisions; El Farol Bar problem, and the minority game, i.e. the existence of situations where the rational (optimal deterministic) strategy is unfeasible or unstable [90]. These put the utility, and maybe the existence, of the *homo oeconomicus* at question. Of course, the assumption of the existence of *homo oeconomicus* has allowed the construction of useful models and theories. The concept of the market being efficient is, for example strictly related to that of rational investors.

Initially, behavioral finance didn't take into account interactions among agents; further developments explored the process of price formation taking the latter as sociological phenomena, hence taking into account interaction between agents (see [81, 135, 82] and references therein). Behavioral finance aims at relating individual decision-making and individual interactions to the emergence of financial structures. The ultimate scope is to be able to explain aggregate anomalies, such as the excess volatility of returns, or volatility clustering, or excess Kurtosis [13], or the equity premium puzzle and the closed end puzzle [127] or other systematic stylized facts in the distribution of returns across markets and financial instruments. Going beyond the expected utility theory allows the use of a different measure of *risk aversion*. [10] proposes an interesting example of an experimental approach in exploring the relationship between financial news and financial decisions. For a review and an assessment of the literature on neoclassical vs. behavioral finance see [127].

Beyond equilibrium I: the role of disequilibrium. Walrasian general equilibrium models are based on the assumption of many representative agents with olimpic rationality that achieve the Pareto optimum equilibrium, in other the state of the economy where the utility of an agent cannot be improved without reducing that of another individual. The social interaction summarized in markets is perfectly efficient, where the Walras equilibrium is associated to Pareto optimum transactions. In the Arrow-Debreu approach, where the case of multivariate stochastic process is under analysis, the market equilibrium is described as the process such that markets clear at each point in time. As above, it is worth mentioning the assumption that the economy tend to a unique and stable equilibrium is very convenient for the mathematical treatment of an otherwise complex analysis. However we ask whether, and eventually how much, this assumption is realistic in modeling real economies within an acceptable margin of error. Time and uncertainty are fundamental part of the modeling problem, when financial decisions are at hand. Uncertainty means that agents that are not perfectly informed may not be able to rely on realistic probability distributions related to future events.

Beginning with Prigogine [117], out-of-equilibrium economic systems come into play. Markets that operate in non-stationary disequilibria are also in Mandelbrot [107]. Dynamic markets models determine non-stationary distributions, whose moments are non constant. Just to give a very simple example, consider the evaluation of the efficient frontier in a portfolio. The efficient frontier in the space of mean-variance is based on the evaluation of the constant correlations among assets [108]. However, in financial markets correlations are not necessary stable, because of the evolutionary intrinsic character of the market, as correlations depend ultimately on human behavior. The idea out-of-equilibrium economic systems is strictly related to the basic principles of the Complexity Theory [18, 19]. As remarked by Arthur, the uncertainty effect is even worse when other agents are involved, thus becoming self-reinforcing and then "non equilibrium arises endogenously in the economy". It is anyway assumed that equilibrium may remain a useful first-order approximation. Last but not least, we discuss the role of technological innovation. As Acemoglu-Robinson [4, 5], technological innovation, intended in a broad sense, introduces *turbulence* in the society, which may produce disequilibrium effects [76]. The introduction of technological innovation by the incumbent ruler may reduce the *political fitness* of the same incumbent ruler.

Beyond equilibrium II: endogenously evolving networks. Another open challenge to the mathematical modelization of social systems is linked to the different levels at which interaction take place, namely whether at the agents or at the macroscopic level. The different levels of interaction correlate, in turn, to the spontaneous formation of self-organized structures at different layers of the hierarchical system. The literature emphasizes how analyzing the links connecting economies with each other, in order to appreciate the systemic dynamic of an economy in the distribution of economies [118]. Moreover, the action of the economic policy at different levels (from micro to meso, and from meso to macro) is proposed by Gallegati [83] in order to go beyond both the Keynesian economic policy and that neoclassical. For instance, [84] shows how the wealth distribution is crucially depending on the underlying interaction network.

Beyond linearity: the role of nonlinear interactions. Emergent properties are those that "cannot be derived analytically from lower-level constructs" [88]. Under a reductionist approach, the aggregate solution is obtained as the sum of the choices each agent makes. This rules out any potential emergent property. In this modeling scenario "there is no difference between micro and macro: the dynamics of the whole is nothing but a summation of the dynamics of its components" [83]. If nonlinear interaction is allowed, the behavior of the system as a whole is not necessarily the sum of the behavior of the single agents. Each agent has an influence on the system which, in turn, acts upon the agents via a *feedback effect* [90], sometimes called the autocatalytic effect [83]. Moreover, when the nonlinearity enters the picture, the proportionality between cause and effect is at question, so that small changes may produce large outcomes: the phenomenon also called the *Black Swan* event [38]. Also, *phase transitions* may be obtained as self-organization phenomena of nonlinearly interacting agents [77].

Beyond numerical simulation: the role of computation. Simulations are the base of the so called agent-based models. The idea is that of simulating the dynamics and the outcomes of agents who reflect the main features of *complex adaptive systems*, including the fact that they interact also with nonlinear features, they learn and adapt. The agent-based modeling focuses on the individual level. For a list of modeling software and a review of the formalism, see [110]. Another viewpoint consists in using numerical tools and analysis to validate and refine theoretical models. One point in the validation of models consists in verifying that they describe quantitative results delivered in quasi steady states corresponding to experiments as an output of the dynamics at the micro-scale, without artificially inserting them into the model, e.g. as a trend to equilibrium. Another point consists in verifying that they describe, at least at the qualitative level, collective behaviors both if far from the steady state, or if in it. Moreover, data should be used toward the assessment of models at the micro-scale. In [9] a method is proposed to use *big data* for predictive purpose, or for model validation. *Big data* opens new possibilities in the field of quantitative understanding of complex social systems. Empirical data, for instance, may be retrieved from opinion pools or elections in the case of opinion dynamics or market analysis in the case of financial or economic models. An entire chapter (ch.13) of the Helbing's book [90] is devoted to the determination of the model parameters from empirical data: once the model parameters are evaluated usually with the method of least squares, a sensitivity analysis should be carried out; a decomposition with respect to explanatory variables can be done using a regression analysis in connection with a factor analysis or a ranking regression analysis.

Another suggestion that is worth mentioning is the possibility to empirically individuate the subsystem to use in a network by means of the *cluster analysis* [90]. Indeed, model validation may find difficulties, for instance related to the fact that the experimental observations may be at a different scale with respect to the scale of the model. For an important review on the so called Computational Social Science see [66]. Finally we observe that, because of the complexity of some models, a complete analytical closed form solution cannot be achieved; in this case computation may be a way of studying the system outcomes [19]. Interestingly, computer is defined as an "exploratory lab" for economics, and, used skillfully, a powerful generator for theory.

3. Critical survey on the interaction between mathematics and socio-economic sciences

This section presents a critical overview of a variety of mathematical approaches known in the literature. Namely, population dynamics, population dynamics with internal structure, game theory, evolutive games, mean field games, statistical dynamics, kinetic theory. This overview should also discuss advantages and withdraws of the different tools and explain why the kinetic theory approach has been selected according to the aims of our paper.

The sequential steps of our presentation are proposed in the next subsections devoted to the following topics:

(i) Extracting from the phenomenological description proposed in Section 2 a number of key features that should be necessarily taken into account in a mathematical approach;

(ii) An overview of the mathematical methods that are known in the literature to model the class of systems under consideration;

(iii) Motivations of the selection of kinetic theory methods as the preferred mathematical tool and critical analysis toward further developments.

Therefore, this section can be viewed as a bridge from Sec. 2 to Sec. 4, where mathematical tools are presented in view of their application to the modeling of social and economic sciences. In more details it is shown how the heuristic description proposed in Section 2 can be taken into account, at least partially, by mathematical tools. Moreover, it is shown how further developments are needed to obtain a full achievement of the said description. This topic will be made more precise in Section 4. In fact the description here is proposed still at a descriptive level with the aim of promoting an interdisciplinary approach, while the mathematical formalization is postponed to Section 4. However, a necessary remarks is that the visionary ideas presented in Section 2 have not yet exhaustively treated in the literature. Therefore, the critical analysis proposed in Section 3 and 4 will enlighten what has been done as well as what should be done. This critical analysis is introduced in this section, but it is more extensively treated in Section 4.

3.1. Key features of socio-economic systems

We consider a large system of interacting entities which can be aggregated into groups of interest. The overall system is behavioral as all living, hence complex, systems. Their common feature, which is independent of the mathematical approach that is used to model them, is that their description needs, at least in principles, a large number of variables. Hence the mathematical approach might give rise to a huge system of equations such to be technically impracticable. Therefore, one of the tasks of the interplay between mathematics and soft sciences in general consists in reducing the aforementioned technical complexity, although preserving the main features of each system under consideration. This topic is treated in [9] focusing on social systems, where the authors argue that a general approach for the modeling of all living systems can be derived. We do agree with this view point, although we argue that it ought to be specialized for each class of systems. Accordingly, we have selected a number of key features to be accounted for in the mathematical modeling tools.

1. **Ability to express a strategy and heterogeneity:** *Individual entities have the ability to express a specific individual strategy. Individuals can aggregate into groups of interest which follow a common strategy and might develop a collective intelligence. The said strategy is expressed with heterogeneous levels within each group of interest.*
2. **Nonlinear interactions:** *Interactions are nonlinearly additive, generally are nonlocal in space, and follow rules that evolve over time and include a continuous adaptation to the changing-in-time environmental conditions.*
3. **Learning and adaptation:** *Individual entities have the ability to learn from past experience. This dynamics can occur between individual entities, individuals and groups and between groups.*
4. **Selection and evolution:** *Interactions include the formation of groups of interest, which in turn can generate new groups more suited to an evolving social and economic environment. Selection is modeled by interactions between the internal and external systems.*

3.2. A survey of mathematical tools

Different mathematical approaches have been used to model the class of systems under consideration. Let us overview some of them, however without claim of completeness. An introductory bibliography is given for each method, while further bibliography is given in the next sections, focusing on specific applications.

• **Methods of population dynamics** can be developed by subdividing the overall systems into different interacting populations. The models study how the number of individuals in each population changes under the action of interactions and environmental processes. Models are classically stated in terms of systems of ordinary differential equations, whose unknowns are the sizes of the various populations.

Heterogeneity can be introduced by selecting populations featured by different behaviors. A dynamics across populations can be properly modeled as it is shown in paper [85]. A widely applied development is offered by the study of *populations with internal structure* well settled in the mathematical framework in [138], formalized through systems of partial differential equations, while various applications are presented in the book [114]. This approach introduces an additional variable, besides the number of individuals, describing inner characteristics of the population, which is supposed to play a role in the emergence of collective behaviors (for instance, the age of the individuals, their fitness for the outer environment, their social status, etcetera).

• **Theoretical tools of game theory.** An important contribution to understand the dynamics of the systems under consideration is given by the theory of evolutionary games [111], see also [123, 124] and the bibliography cited therein. These new theories study how players' strategy evolves in time due to selective processes, which can lead to clustering of the players into different groups depending on their fitness for the outer environment. The time evolution of game dynamics can be modeled in terms of differential games [53], where players apply a control action over basic dynamics modeled by ordinary differential equations in order to increase their payoffs.

A further development of evolutionary game theory is given by the so called *mean field games* introduced in [103, 104, 102] and subsequently applied by several authors to model financial markets and socio-economic systems in general (see [7] for recent developments). This theory aims at modeling and analyzing complex decision processes involving a large number of indistinguishable rational agents who have individually a very small influence on the overall system and are, on the other hand, influenced by the mass of the other agents. Mean field equations can be derived to replace the many particles interactions by a single problem with an appropriately chosen external mean field which takes into account the global behavior of the individuals.

• **Stochastic games and learning** look more precisely at large social systems. This approach has been reviewed and critically analyzed in [9], while a concise description has been given in [35] as follows:

1. The overall state of the interacting entities, which are viewed as active particles, is delivered by a probability distribution function over the state, at the microscopic scale, of the particles. Such micro-state includes social variables in addition to the mechanical variables such as position and velocity.
2. Active particles have a sensitivity domain and they interact with all particles included, at each time, in such domain. Interactions can also be nonsymmetric with respect to the microscopic variable, while the output of the interaction of stochastic players is also given in probability.
3. Interactions are nonlinearly additive as the output depends not only on the microstates of the interacting particles, but also on the aforementioned probability distributions, namely the dependent variable of the differential model.

These theoretical tools show a direct connection with the mathematical theory of active particles which is introduced in the following, while have been recently developed toward a modeling approach to collective learning dynamics [57, 58].

• **Methods of statistical mechanics and of the kinetic theory's approach** have been developed by Helbing, who understood that individual interactions need to be modeled by methods of game theory. A unified approach, suitable to link methods of the statistical mechanics, kinetic theory, and game theory, is proposed in the book [90], where master equations are derived to provide a quantitative information on the dynamics of a variety of social systems. A general purpose of the activity by Helbing and coworkers aims at modeling the overall dynamics of modern societies, where hard sciences such as mathematics and physics can contribute to improve the quality of life [91].

A parallel approach is that of the kinetic theory presented in [113], which uses Boltzmann-type equations where the velocity is replaced by internal variables related to the specific social systems under consideration. Nowadays, the book [113] appears to be the most important reference in the field as it provides an exhaustive overview covering the whole path from numerical methods to mathematical tools and applications, such as opinion formation and wealth dynamics, as well as various aspects of the social and economic dynamics of our society.

A more specialized framework of the generalized kinetic theory is given by the Fokker-Plank approach which has been applied by various authors to modeling social sciences. The recent paper [79] is an interesting example of this type of applications, while it provides an updated bibliography in the field.

• **The kinetic theory of active particles** corresponds to the formalization of the five issues of our rationale and it is presented in Sec. 4, where it is shown that the individual entities are called *active particles*, the overall system is subdivided into group of interest called *functional subsystems*, while the strategy expressed by the active particles is depicted by an internal variable at the microscopic state called *activity* which can differ from particle to particle. Particles that express, although heterogeneously, the same activity, are grouped into the same functional subsystem. Interactions are modeled by theoretical tools of game theory. Finally, the overall state of the system is described by a probability distribution over the microscopic state. The development of this approach to modeling social systems has been introduced in [8] and subsequently developed, through various application, to a systems approach presented in [9]. Motivations on the use of this approach are given in the next subsection.

3.3. Motivations toward kinetic theory methods

The need to include the main features of socio-economic system in the modeling approach imposes the search of more sophisticated tools to be used in the mathematical structure. The recent literature [8, 9] shows that such behavioral features can be taken into account by suitable developments of the method of the kinetic theory for active particles, which provides a general mathematical structure appropriate to offer a conceptual framework for the derivation of specific models, and presents, according to the authors' opinion, the most appropriate approach for the modeling of behavioral systems constituted by a large number of living entities. This selection is based on the idea that the approach should be flexible enough to capture the specific features reported in Section 2 for systems, where a behavioral dynamics, which is typical of all living systems, plays an important role. Indeed, we do not naively claim that the approach provides an exhaustive reply to all aforementioned queries. In fact some important issues have not been exhaustively treated, while various problems appear to be still open. Our paper aims at bringing to the attention of interested researchers these problems.

The rationale behind this choice, stems from the idea that the development of the methods of the kinetic theory by including more features of living system, can provide a systems approach, and hence a field theory, for the modeling of socio-economic systems, which is not only capable to reduce the large number of variables provided for the description of a complex living system but also is able to capture the main complexities presented in Subsection 3.1. This theory, as it is presented in Sec. 4, already offers a formal framework deemed to capture the complexity features of large living systems. In more detail:

- ***Ability to express a strategy and heterogeneity:*** *The microscopic state includes the activity variable, which is deemed to model the strategy expressed by each functional subsystem, while the heterogeneous behavior is accounted for by the heterogeneity of the activity variable within the active particles and by the description of the system by means of a probability distribution linked to each functional subsystem.*

- ***Nonlinear interactions:*** *The mathematical structure of Sec. 4, includes nonlinearly additive interactions, where nonlinearity refers to the structural nonlinearity due to the products of the distribution functions, as well as to the output of the interactions which is conditioned not only by the state of the interacting entities, but also by the probability distribution over such states.*

- ***Learning and adaptation:*** *The ability of living systems of learning from past experiences is accounted for in the modeling of interactions based on rules that evolve over time, and that include a continuous adaptation to the changing-in-time environmental conditions.*

- ***Selection and evolution:*** *The onset of functional subsystems is modeled by the transition probability density, where interactions include the formation of groups of interest, which in turn can generate new groups more suited to an evolving social and economic environment.*

In addition, the functional systems approach selects those parts of the overall system, which play effectively the role. Therefore, the selection of the dependent variables related to each functional subsystem accounts also of the need to reduce complexity induced by the generally large number of components.

As a critical analysis to the motivations proposed in this subsection we anticipate, waiting for a more exhaustive treatment in the next section, that one of the problem left open among those presented in Section 2 is the modeling of the formation and dynamics of endogenous networks. Indeed it is a challenging problem posed in the book [51], but not yet exhaustively treated in the literature. Another problem, worth to be mentioned, is the interaction of different types of dynamics that corresponds to several socio-economical systems. The next sections will devote some space to these two research perspectives.

4. Mathematical tools

This section provides a technical presentation of the tools of the kinetic theory of active particles focused on a socio-economic system constituted by a large number of living entities interacting in a territory. These entities can be grouped into different aggregations, for instance *groups of interest*, whilst their specific features are heterogeneously distributed within each group.

The contents are proposed within a formalized framework which refers to the existing literature, where the main sources are [9] and [33]. The rationale of the mathematical methods is explained and critically analyzed so that the contents of this section are communicated to a broad audience and not simply to devoted mathematicians. This framework provides the conceptual basis for the applications reviewed in the next section.

4.1. Rationale of the kinetic theory of active particles

The method of the kinetic theory for active particles provides a general mathematical structure suitable to capture the main complexity features presented in Sec. 3 and to provide the conceptual framework for the derivation of mathematical models. This framework is the core of a systems approach to social dynamics which was initiated in [8] and subsequently developed through various papers [9, 38, 59, 85].

The derivation process and the role of such a mathematical structure in the modeling approach is depicted in Fig. 1, which summarizes the following sequential steps:

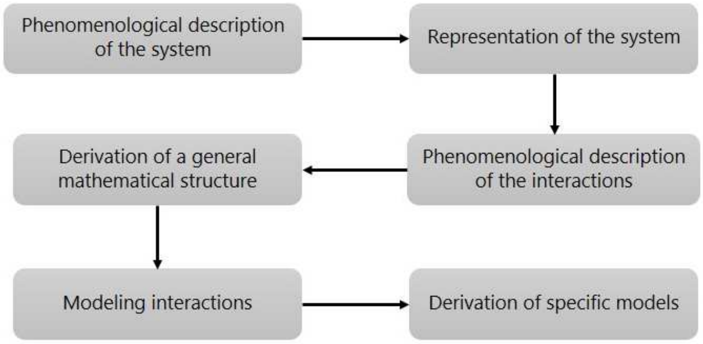


Figure 1: Sketch of the major steps of the Kinetic Theory for Active Particles (KTAP)

1) **Phenomenological description:** The entities that comprise the system are referred to as *active particles* and are aggregated into different groups of interest called *functional subsystems* (FS), labeled by the subscript $i = 1, \ldots, n$. Active particles within the same FS share a common strategy called *activity* which define their *microscopic state*.

2) **Representation:** At the *microscopic scale* the state of the system is delivered by a *probability distribution*, linked to each functional subsystem, over the activity variable, while the *macroscopic* description is provided by the *statistical moments* related to the said probability function.

3) **Description of interactions:** Interactions include encounters between active particles as well as between particles and functional subsystems. In these interactions, the FS is viewed as a whole being represented by the mean activity value of the FS.

4) **Derivation of the mathematical structure**: The *differential structure* which describes the evolution of the probability functions, is related to the interactions and it is obtained by a balance between the inflow and outflow of particles within the elementary volumes of the space of microscopic states.

5) **Modeling interactions:** Theoretical tools of *stochastic behavioral games* [63, 87, 90, 111] and the tools provided by *individual/collective learning theory* [57, 59], contribute to the conceptual framework for the modeling of the interactions at the microscopic scale.

6) **Derivation of models:** Specific models are obtained by implementing the aforementioned modeling of interactions in the general mathematical structure derived in item 4).

The validation of models follows by comparisons with empirical data and it is required to reproduce quantitatively empirical data whenever available. In this case, a model shows predictive ability. However, models can also be used for exploratory purposes by showing different trends according to different values of the parameters. In some cases, a model can show trends that cannot be predicted by heuristic reasoning, namely rare events, occasionally defined "black swans" [131].

As mentioned above, the state of each functional subsystem is defined by a probability distribution over the activity variable:

$$f_i(t,u) \; : \quad [0,T] \times D_u \;\rightarrow\; \mathbb{R}_+, \tag{1}$$

where the subscript i denotes the functional subsystem, and the domain of definition of the activity variable, D_u, is assumed to be a symmetric bounded interval of $\mathbb{R}$, typically $u \in [-1, 1]$. This representation correspond to the description in item (1) at the beginning of this subsection, namely the subscript i labels the groups of interest, while the variable u corresponds to the strategy developed by each group. The domain $[-1, 1]$ corresponds to a technical choice, where $u = 0$ denotes null expression, while $u = -1$ and $u = 1$ denote, the maximal values, respectively, positive and negative of such expression. The distribution functions are nonnegative and can be scaled with respect to the total number of active particles at the initial time. This simplification of physical reality amounts to admitting that "soft" variables cannot be measured precisely as most of the physical variables. Hence we can assess a lower and upper bound for each variable and assume their continuous spanning within such domain.

In this way, under suitable integrability conditions, $f_i(t,u)du$ gives the number of active particles of the i-th FS that are, at time t, in the elementary volume $[u, u + du]$ of the microscopic space D_u. The description of the system at the macroscopic scale is obtained by implementing standard moment calculations. For instance, the 0-th order moment gives the number density

$$n_i[f_i](t) = \mathbb{E}_i^0[f_i](t) = \int_{D_u} f_i(t,u)\, du, \tag{2}$$

which provides the number of active particles in the i-th functional subsystem, while higher p-order moments are obtained as follows:

$$\mathbb{E}_i^p[f_i](t) = \frac{1}{n_i[f_i](t)} \int_{D_u} u^p\, f_i(t,u)\, du. \tag{3}$$

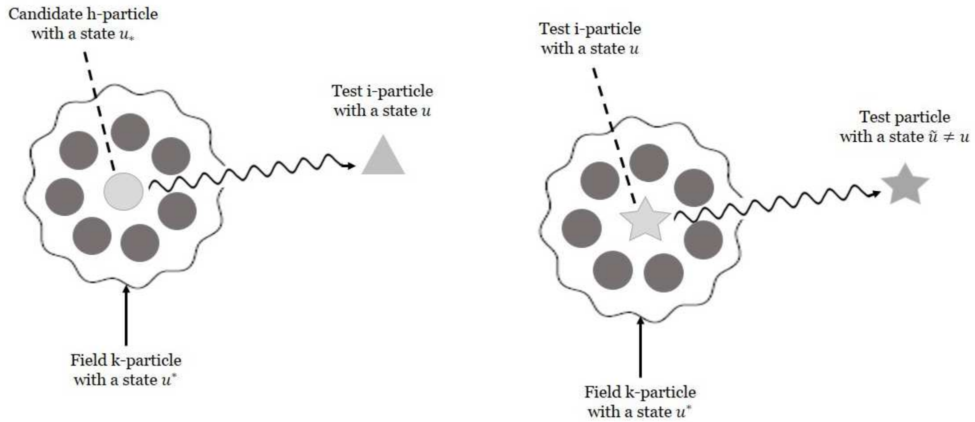
In particular, the first-order moment correspond to the mean activity of the i-th FS, while second-order moments can be referred to some equivalent energy related to the activity variable.

As it is mentioned, the derivation of the evolution equation for each distribution function $f_i(t,u)$ is related to the interactions at the microscopic scale. Moreover, the description of the interactions among active particles requires to distinguish three types of particles, namely:

- The **test** particle of the i-th FS (i-particle for short) with activity u, is the representative entity of the system;
- The **candidate** h-particle with activity u_*, is the particle which can gain the test state as a consequence of the interactions;
- The **field** k-particle with activity u^*, is the particle which triggers the interactions of candidate particles.

The derivation of the mathematical structure refers to two types of interactions:

- **Micro-micro scale interactions:** Individual interaction within the same FS or across FSs between active particles;



(a) Inlet micro-micro interactions: the candidate h-particle with activity u_* gains the state u of the test i-particles, which are in its sensitivity domain, after an interaction with a field k-particle with a state u^*.

(b) Outlet micro-micro interactions: the test i-particle with activity u loses its state after an interaction with a field k-particle with activity u^*.

Figure 2: Micro - micro scale interactions: interactions between particles in which the test particle loses its state or other particles gain it.

- **Micro-macro scale interactions:** Interactions between particles and their FS viewed as a whole being represented by the mean activity of the FS.

Micro-micro scale interactions are visualized in Figures 2, while a similar illustration can be used in the case of micro-macro interactions. Moreover, for both types of interactions we consider the following classifications:

- **Inlet interactions:** They refer to the interactions in which the candidate particle loses its state and adopts the state of the test particle;
- **Outlet interactions:** Interactions by which the test particle loses its state after an interaction with field particles or a whole FS.

The dynamics of the inlet and outlet interactions are visualized, respectively, in Figures 2(a) and 2(b).

It is worth anticipating two issues that will be discussed more exhaustively in Section 6. Namely the use of a scalar representation for the activity variable u is based on the assumption that the subdivision into functional subsystems leads to a description, where each subsystem ultimately express a scalar activity. However, in various cases it might be using a vector variable. In addition, the activity variable is the micro-scale variable which, however, might include, in some cases, additional variables such as position and velocity. The literature on this topic, requires additional work to achieve an exhaustive treatise. Therefore, we will return to this topic in the last section focusing on research perspectives.

4.2. Derivation of a mathematical structure

Bearing all above types of interactions in mind, the dynamics of active particles in an elementary microscopic volume can be described by the following rule:

Time variation of the number of active particles
= Inlet flux due to micro-micro interactions
− Outlet flux due to micro-micro interactions
+ Inlet flux due to micro-macro interactions
− Outlet flux due to micro-macro interactions.

Keeping in mind that the inlet and outlet flux are related to the inlet and outlet interactions respectively, one is led to the following mathematical structure [9]:

$$
\begin{aligned}
\partial_t f_i(t,u) &= \sum_{h,k=1}^{n} \int_{D_u \times D_u} \eta_{h,k}[\mathbf{f}](u_*, u^*)\, \mathcal{A}^i_{h,k}[\mathbf{f}](u_* \to u | u_*, u^*) f_h(t,u_*)\, f_k(t,u^*)\, du_*\, du^* \\
&- f_i(t,u) \sum_{k=1}^{n} \int_{D_u} \eta_{i,k}[\mathbf{f}](u, u^*)\, f_k(t,u^*)\, du^*, \\
&+ \sum_{h,k=1}^{n} \int_{D_u} \nu_{h,k}[\mathbf{f}](u_*, \mathbb{E}^1_k)\, \mathcal{B}^i_{h,k}[\mathbf{f}](u_* \to u | u_*, \mathbb{E}^1_k[f_k]) f_h(t,u_*)\, du_* \\
&- f_i(t,u) \sum_{k=1}^{n} \nu_{i,k}[\mathbf{f}](u, \mathbb{E}^1_k),
\end{aligned}
\tag{4}
$$

where $\mathbf{f}$ denotes the set of all distribution functions. In Eq. (4) the square brackets have been used to denote the dependence on the distribution functions which highlight the nonlinear nature of interactions.

The quantities appearing in Eq. (4) which model interactions are

- The *interaction rates* $\eta_{h,k}[\mathbf{f}](u_*, u^*)$ and $\nu_{h,k}[\mathbf{f}](u_*, \mathbb{E}^1_k)$: which model the frequency of interactions between a candidate h-particle with state u_* and a field k-particle with state u^* or a k-th FS viewed as a whole being represented by its mean activity $\mathbb{E}^1_k$;

- The *transition probability density* $\mathcal{A}^i_{hk}[\mathbf{f}](u_* \to u | u_*, u^*)$: which denotes the probability density that a candidate h-particle shifts, after an interaction with a field k-particle, to the state of the test i-particle, such that the microscopic states of the candidate h-particle and the field k-particle are u_* and u^* respectively;

- The *transition probability density* $\mathcal{B}^i_{h,k}[\mathbf{f}](u_* \to u | u_*, \mathbb{E}^1_k)$: which denotes the probability density that a candidate h-particle shifts, after an interaction with the k-th FS, to the state of the test i-particle, such that the microscopic state of the candidate h-particle is u_* and that the mean activity of the k-th FS is $\mathbb{E}^1_k$.

A particularized structure can be obtained when the dynamics across functional subsystems is not taken into account:

$$
\begin{aligned}
\partial_t f_i(t,u) \quad = \quad & \sum_{k=1}^{n} \int_{D_u \times D_u} \eta_{i,k}[\mathbf{f}](u_*, u^*)\, \mathcal{A}_{i,k}[\mathbf{f}](u_* \to u | u_*, u^*) f_i(t,u_*)\, f_k(t,u^*)\, du_*\, du^* \\
- \quad & f_i(t,u) \sum_{k=1}^{n} \int_{D_u} \eta_{i,k}[\mathbf{f}](u,u^*)\, f_k(t,u^*)\, du^*, \\
+ \quad & \sum_{k=1}^{n} \int_{D_u} \nu_{i,k}[\mathbf{f}](u_*, \mathbb{E}^1_k)\, \mathcal{B}_{i,k}[\mathbf{f}](u_* \to u | u_*, \mathbb{E}^1_k[f_k]) f_k(t,u_*)\, du_* \\
- \quad & f_i(t,u) \sum_{k=1}^{n} \nu_{i,k}[\mathbf{f}](u, \mathbb{E}^1_k),
\end{aligned}
\tag{5}
$$

Mathematical models can be obtained by inserting into Eq. (4) or Eq. (5) specific models of interactions at the microscopic scale, namely the encounter rates $\eta_{h,k}$, $\nu_{h,k}$ and the transition probabilities $\mathcal{A}^i_{hk}$, $\mathcal{B}^i_{h,k}$ need to be specified. The structure indicates that these quantities can depend on $\mathbf{f}$, where the square brackets are used to distinguish nonlinearity in the parameters from the structural nonlinearity due to the products of the distribution functions.

Some remarks which enlighten specific properties of this mathematical structure, as well as a rationale toward the modeling of the interaction terms, are brought to the attention of the reader:

- The mathematical structure presented in this section deals only with the case of closed system in which the number of particles is preserved; this assumption is reasonable only if the time scale of observation and modeling of the system itself is not long enough to observe the birth and death events as well as the inlets from the outer environment. However, this structure provides the framework for the derivation of the class of models presented in this paper. For a technical generalization of a such structure to include proliferative/destructive interactions as well as external forces the reader is referred to [33].

- The frequency of the interactions among candidate and field particles is, in general, assumed to decay with the distance between the interacting particles, where an exponential or a polynomial decay, i.e a decay described by rational fractions, can be adopted as the most appropriate. In the linear case, the distance between the particles involves only the microscopic states u_* and u^* of the candidate and field particles, respectively, and can be defined as $|u_* - u^*|$, which is referred as *microstate-microstate distance*. Conversely, nonlinear interactions occur when an *affinity distance* is used between active particles characterized by different distribution functions and can be defined by $\|f_h - f_k\|$, where $\|\cdot\|$ is a suitable norm, to be selected depending on the characteristics of the system and in the specific investigation under consideration. In the case of micro-macro interactions the distance involves the microscopic state u_* of the candidate particle and mean value of the activity $\mathbb{E}^1$ of the group of particles belonging to a functional subsystem, and can be defined as $|u_* - \mathbb{E}^1|$, which is referred as *micro-macro distance*.

- The probability densities $\mathcal{A}^i_{hk}$ and $\mathcal{B}^i_{h,k}$ can be modeled by relating the payoffs of the interactions by theoretical tools of game theory. Various types of possible games have been introduced in [9, 33] and can be summarized as follows: *cooperation (consensus) games*, where particles tend to share their microscopic states by decreasing the difference between their states; *learning dynamics*, where one of the two particles modifies, independently of the other, its microscopic state, while the other reduces the distance by a learning process; *hiding/chasing dynamics*: one of the two particles attempts to increase its distance from the state of the other one (hiding), which conversely tries to reduce it (chasing). In general, different types of games occur in the interactions. Their occurrence, namely the prevalence of one type with respect to the other, is ruled by a threshold on the distance between the states of interacting particles, however, such a threshold can depend on the state of the system as a whole and that can have an important influence on the overall dynamics [38, 74]. Such a threshold is, in the simplest case, a constant value. However, recent papers [38, 74] have shown that the threshold can depend on the state of the system as a whole and that can have an important influence on the overall dynamics.

- The splitting into functional subsystems has not a universal meaning. In fact, it depends upon the specific investigation under consideration. Therefore, different modeling perspectives correspond to different strategies to decompose the system. However, the functional system approach differs from classical approaches where the overall systems is featured by physical components well localized in the environment. Here, each subsystem is featured by the specific functions expressed in the specific dynamics under consideration.

- The mathematical frameworks characterized by Eqs. (4) and (5) have been derived under the assumption that the activity variable is continuous over the domain D_u. Some authors have used the same framework extended to the case of variables that attain a discrete number of values within D_u. This is a technical choice which does not affect the reasonings presented in the sequel. So that we will refer to Eq. (4) and (5) also in the case of discrete activity variables. The interaction domain D_u which is assumed, for simplicity, constant. However, a dependence of D_u on the distribution function should be taken into account. A possible example has been proposed in [40] for the modeling of swarm dynamics, in which the interactions occur in a sensitivity domain $\Omega \subseteq D_u$ defined so as to contain a critical number of field particles.

Let us now consider two features, the first one might be used for validation of models, while the second one to further possible developments. Validation of models is based not only on their ability to depict empirical data obtained in steady cases, but also to reproduce emergent behaviors that appear far from steady and equilibrium conditions. Indeed, in several cases collective emerging behaviors are reproduced at a qualitative level given certain input conditions, through quantitative matches. In addition, a model might even foresee rare events, namely the so-called *Black Swans* [131], which cannot easily be predicted, but when these events are observed the causes that have generated it are rapidly identified. In particular, models should be required to contribute to detect early signals of this type of event [126].

An additional remark is that living entities can express a large variety of activities. However, due to the need of reducing the technical complexity induced by an excessive number of equations, we restricted the modeling approach to those specific activities that are object of study. Moreover, we consider, in the above formalization, only cases where the activity within each functional subsystem is a scalar. On the other hand, the use of vector activity variables is needed when the system is subject to the interaction of different type of dynamics. An example is given in [38, 76] where the interplay between wealth distribution and the support or opposition to a Government is accounted for and in [76], described in the Case Study II of the present paper. Further analysis on this topic is proposed in the last section of this paper.

Before moving to the next section, where some case studies will be presented, we wish mentioning that the mathematical structures delivered by Eqs. (4) or (5) generate models with high nonlinearity, namely not only in the quadratic terms over the dependent variables f, but also in the parameters that are allowed to depend on f. Therefore, the search of analytic solutions is restricted to a very few special cases. Hence computational methods are necessary. The applications known in the literature have widely applied the so called particle Monte Carlo methods introduced by Bird [47] and subsequently developed by various authors [15, 25, 113]. The selection of the appropriate computational tools should also take into account which type of response is expected by simulations. In this specific case, it is interesting, and useful for socio-economical systems, obtaining the whole distribution function and not only the low order moments as it occurs in fluid dynamics when one moves from molecular to continuum. The development of Monte Carlo methods starts from the modeling of interactions as a first step to develop a code. This approach has been applied to solve models of crowd dynamics [37], while a systematic development suitable to include all admissible interactions would definitely be an interesting, however challenging, research perspective.

5. Applications to socio-economic questions: two case studies

This section reviews some of the results obtained by applying the kinetic theory methods to questions raised in socio-economics. More specifically, we present an overview of the literature on the developments and applications of the kinetic theory approach, and two case studies aiming at showing applications of the methodology under discussion, respectively.

5.1. Kinetic theory: developments and applications to social science

Methods drawn from the kinetic theory have been applied in several fields of social sciences. Opinion formation is an example where these methodologies are adopted (among others, see [12, 48, 52, 54, 78, 89, 113, 133], together with questions raised when modeling behavior of financial markets is of interest [17, 67, 68]. Kinetic equations with game theory type interactions was first introduced in [16], and then in [43], that extended the analysis to modeling social systems with discrete microscopic states. In this analysis, a system of ordinary differential equations is derived to describe the time dynamics of a discrete probability distribution corresponding to the discrete micro-states. A very similar approach, with minor technical differences, is extended to various types of applications focusing, for example, upon opinion formation [45], taxation [44], and value estimate with public information and herding [72]. Other developments has followed the above pioneering research. Various authors propose alternative approaches to the systems theory of social dynamics (for instance, see [23, 51]). The first contribution within the framework of the kinetic theory approach is in [8], where the concepts of scaling and functional subsystems are proposed for a variety of social systems, highlighting the need of modeling the interactions of different types of dynamics. Nonlinearity for interactions at the micro-scale is in [38] and [74]. Building on previous literature [115], [86, 100, 101] discuss social aspects of selfishness and wealth distribution. The dynamics of mutations due to selection plays an important role in biology, specifically in immune competition [41, 70]. An analogous dynamics in socioeconomic systems is due to new groups of interest, that emerge because of for example the aggregation of different groups which subsequently may either expand or disappear in a competition mediated by the environment [70]. Surprisingly, economics has been somewhat left apart, especially if compared to other social sciences. We present two case studies regarding applications of the kinetic approach to economics.

5.2. Case study I: Wealth redistribution

We briefly summarize [74], that proposes a model of wealth redistribution. We discuss the step by step implementation of the method, and an application of the modeling tools we have discussed in the previous sections. The model treats stylized phenomenologies; however, it shows some interesting qualitative behaviors. More specifically, the model shows that different political scenarios significantly influence the evolution of the distribution of wealth.

- **Socio-economic system:** The paper models the dynamics of wealth redistribution, where the microscopic entities are individuals that interact with each other, given the initial wealth distribution. The paper aims at exploring eventual long-term solutions of the mathematical system drawning on the KTAP theory, which gives the long-term wealth distribution when alternative initial conditions and socio-economic scenarios are considered.
- **Functional subsystems and activity variable:** The society of individuals is not split into different groups of interest, i.e. no subsystems are taken into consideration. The activity variable is the wealth status of the individual, and it is assumed to be a scalar discrete variable that, in turn, produces a corresponding discrete number of wealth classes in the society.
- **Interactions:** Interactions are individual-based. Both the interaction rates and the transition probabilities depend upon macroeconomic quantities. The stochastic output of each interaction depends upon both the wealth status of the interacting individuals and the time-varying mean of the wealth in the society. The dynamics of the exchange is modeled by cooperation/competition games that is triggered by a threshold. In turn this is assumed as being constant or dependent on the macroeconomic quantities, when alternative socio-economic scenario is considered. *In the first of these scenarios an influential government keeps the threshold constant and may or may not determine a greater incidence of co-operation on competition (if the threshold is low with respect to the number of wealth classes introduced). In the second scenario, the dynamics is left to internal competitions and conflicts inside the population and the threshold may vary in time in relation with the evolution of the wealth distribution, determining a time-varying incidence of cooperation on competition.*
- **Emerging behaviors:** A low constant threshold determines the shift of the individuals' wealth classes toward the middle class and, in the long term, a raise of the mean wealth. On the other hand, a large constant threshold determines the polarization toward the lower and the healthy classes and, in the long term, a decreasing mean wealth. Results in the case of a variable threshold are similar. However, *the transients* mean wealth show some differences. The analysis of the influence of the parameter characterizing selfishness (if individual behavior is competitive) shows that a more selfish society determines, in the long run, a reduction of the mean wealth. A more altruistic society (if individual cooperate) leads to higher mean average wealth.

In the following, a scheme of some emergent properties, i.e. properties of the long term wealth distribution, is proposed.

Role of the threshold		
Low constant threshold	⇒	Increasing mean wealth
High constant threshold ∨ Variable threshold	⇒	Decreasing mean wealth

Role of the initial conditions		
Initial condition with prevalent middle class	⇒	Max. positive rate of wealth

Interplay between initial conditions and the threshold				
Prevalent middle class ∨ Prevalent upper class	∧	Const. threshold ∨ Var. threshold	⇒	~ unchanged wealth

Role of the parameter "selfishness"		
"Altruistic" society	⇒	Increasing mean wealth on the long run
"Selfish" society	⇒	Decreasing mean wealth on the long run

5.3. Case study II: Political competition and democracy

This subsection reviews [76], where a model of interactions of different political groups in a a population is modeled. The model refers to a phenomenon based upon historical data and then modeled in the stationary conditions, by Acemoglu and Robinson [4, 5]. In extreme synthesis, the hypothesis is that incumbent rulers tend to block the introduction of technological innovation in the society if the innovation itself raises the probability of being replaced in command, even if the innovation has a positive effect on the citizens' wealth.

This political strategy, defined the "political replacement effect", is likely to emerge if political competition is neither low, as in this case the probability of being replaced in command is low as well, or very high, as in this case the innovation is likely to be introduced by the political competitors of the incumbent. According to the theory, there hence must be a nonmonotonic relationship between the emergence of economic innovations and the level of political competition: the rulers are more likely to introduce innovations, in its broad sense, either if they are highly entrenched, or if political competition is highh. They tend to block innovation where political competition is at intermediate level. [4, 5] discusses historical evidences about this nonmonotonicity, and provides an explanatory model. [105] provides empirical evidence in support of this theory. [76] presents blocking as an emergent property of the society, arising from the micro-micro and micro-macro interactions. Moreover we discuss the conditions under which the nonmonotonic relationship between the introduction of innovations by the rulers and the political competition appears in the context of a dynamic KTAP model.

- **Socio-economic system:** The society of individuals is divided into different groups of interest, each of them expressing different socio-economical purposes and functions.

- **Functional subsystems and activity variable:** There exists three functional subsystems: incumbent rulers, citizens, and political competing group. The activity variable is a vector for each subsystem, namely “political power and propensity to introduce technological innovations” for the incumbent rulers, “wealth and political opinion” for the citizens and, finally, “wealth and political power” for the political competing group.

- **Interactions:**
  Rulers:
  Micro-micro interactions: the rulers’ propensity to innovate is driven by their political power (the higher the political power, the higher the propensity to innovate).
  Micro-macro interactions: the introduction of technological innovation leads to higher the citizens’ wealth as well as higher competing group’s wealth. The political power of the rulers is inversely related to both the mean political power of the competing group and to the mean political opinion of the citizens.

  Citizens:
  Micro-micro interactions: the citizens’ political opinion is assumed to be driven by their wealth.
  Micro-macro interactions: the citizens’ mean opinion influences positively the political power of the rulers.

  Political competing group:
  Micro-micro interactions: the political power of the competing group is assumed to be driven by their wealth.
  Micro-macro interactions: the mean political power of the competing group has a negative impact on the political power of the ruler.

- **Emerging behaviors:**We look at the time evolution of the first moment of the marginal probabilities on the microscopic state. The aim is exploring the role of three different political scenarios on the propensity to innovate of the rulers or on the citizens’ wealth.

  Political scenarios are characterized by different initial distributions on the political power of the rulers and the competing group:

  - I - Strong rulers and weak political competing group (weak democracy): namely rulers owing a high political power and competing with a group with a low political power (wealth distribution is assumed to be uniform in both subsystems.)
  - II - Strong rulers and strong political competing group (healthy democracy); wealth distribution is assumed to be uniform in both subsystems.
  - III - Medium political power both for rulers and political competing group in a poor society and in a rich society, respectively. Namely, in this case the citizens’ marginal distribution on wealth is shifted toward the lowest value or the highest, respectively.

  The evolution of the political power of both the rulers and the competing group as depicted by the model are highlighted in the table below

I - Strong rulers & weak political competing group (weak democracy)		
Initial condition with uniform citizens' wealth & uniform citizens' political opinion	⇒	"Blocking" on innovation ↑ political power of the competing group ↓ political power of rulers.

In the case of this political scenario, the model produces a nonmonotonic relationship between the propensity of the rulers to innovate and the political competition in the society, in support of the hypothesis in Acemoglu and Robinson [4].

II - Strong rulers & strong political competing group (healthy democracy)		
High return of technological innovation on citizens' wealth ∧ High citizens' susceptibility to change opinion	⇒	Society escapes the "trap of blocking"

The term "trap of blocking" refers to the Acemoglu and Robinson's model, associated to the idea that if rulers decide to block innovation, there cannot be a transition to a different equilibrium without blocking.

III - Medium political power both for rulers and political competing group		
Initial condition with poor society	⇒	↓ rulers' political power ↓ competing group's political power
Initial condition with wealthy society	⇒	↑ rulers' political power ↑ competing group's political power

5.4. Critical analysis

The overview of the literature has focused on theoretical issues and applications on the interactions between mathematics and socio-economic sciences, by showing that the reviewed general approach in our paper witnesses a variety of applications in social sciences but a rather limited number of them over economy. However, the literature indicates that the method can model various aspects of the interactions between social dynamics and economics, as well as, reversely, economics and social sciences and the case studies presented in Subsections 5.2 and 5.3 have highlighted this ability. Therefore, possible developments look at the interaction involving different dynamics which involves technical difficulties that will be discussed in the next section.

An interesting feature of the development of the method is that it has evolved according to the needs and suggestions induced by the search of specific models. Therefore, we should expect further developments of the method and the interested reader might hopefully contribute to them.

6. Research perspectives and further developments in economics

This final section looks at potential research perspectives generated by the interaction between "hard" and "soft" sciences. Sec. 4 develops some ideas. Accordingly, contents are devoted to the interactions of different dynamics and to the development of the kinetic theory approach to economics, beginning with the case studies discussed in Sec. 5. A discussion deserves the validation of models, as the literature shows different methodological approaches which are induced by different view points, while the development of a unified approach is expected. Among several possible topics, we have selected the following.

Interactions between social sciences and economics: The general mathematical approach uses, as we have seen, a scalar quantity corresponding to the activity variable. However, it has been remarked in various places, and also in the case study II, that interesting outputs can appear whenever two different dynamics interact. This is a well defined feature of behavioral economics, where individual heterogeneous behaviors derive from social dynamics and the output of social dynamics has an influence on economics or there are two other types of social dynamics ([38, 74, 76]. This research perspective focuses on the interaction between two "soft" sciences. Let us now consider the relatively simple case given by Eq. (5) corresponding to a dynamics without mutations. An abstract approach would simply amount to using a vector variable, as an example $\boldsymbol{u} = \{u, v\}$ and $\mathcal{A}_{i,k}[\mathbf{f}](\boldsymbol{u}_* \to \boldsymbol{u}|\boldsymbol{u}_*, \boldsymbol{u}^*)$. A reasonable a simplification consists in assuming a hierarchy, for instance the dynamics on the activity u drives the dynamics on the activity v, and subsequently factorizing $\mathcal{A}_{i,k}$ as the probability density of the dynamics on u with the probability density of the dynamics on v conditioned by v.

Further developments in economics: A natural question is motivated by the search of a direct application of the general approach to economics without passing through social sciences. Arguably the answer is positive, but one should observe that additional work is needed, as an example by making more precise the scaling problem, such as microscopic to micro-economics and macroscopic to macro-economics. In addition, the structure reported in Eq. (4) should be further developed and investigated. In particular, it would be useful to consider mutations and selections which have not been yet extensively treated in the literature.

A general topic, that indeed refers to any type of interactions, is the modeling of the activity variable which should be a vector suitable to refer to different dynamics. This need is witnessed in various papers starting from [38] to the case study treated in Subsec. 5.3. It is not simply a formal issue consisting in using vectors instead of scalars, but it needs a more sophisticated approach that looks at the hierarchy by which the dynamics can exert an influence one with some priority with respect to the other.

The KTAP approach allows for the analysis of a number of economics-related open questions. Once the theoretical framework is designed, the challenge consists of adopting the kinetic approach to interpret real-world economics issues. In the following we propose a research issue that, in the authors opinion, has the potential of being addressed along this perspective, and it is based upon macroeconomic data.

Based upon the GDP (Gross Domestic Product) per capita and the Technology (T) for a sample of world economies, we estimate and study the evolution of their bivariate distribution. According to the KTAP modeling framework, world economies act as active particles where economies are described by a vectorial microstate with two components, given by GDP and T, respectively. The aim is to study the conditions under which we observe either economic absolute convergence, that is the shift of the whole world economies toward the status of leader economies, or club-convergence instead, that is convergence within few alternative subgroups. We consider different subsystems, to account for the heterogeneous behavior of the economies and to explore the impact of this heterogeneity on the long run economic convergence toward leader countries.

The idea is to consider a subsystem of leader economic countries together with few subsystems of follower countries, clusterized according to a rank which takes into consideration the so called social capability, defined as the socio-cultural environment influencing the responses of a society to economic opportunities, such us the imitation of more efficient technology, as in the case we want to study cite[A86]. Whether economic absolute convergence or club convergence across economies emerge is likely to be due to interaction interaction across economies [31]. In the model the time evolution of the bivariate distribution of world economies over the GDP and T may be driven by both micro-micro and micro-macro interactions. Micro-micro interactions could be modeled according to stochastic competitive games, depending on whether competitive binary interactions with economies belonging to the same functional subsystem make each economy richer or poorer. The per-capita GDP may also depends upon the mean level of T of the functional subsystem into consideration. Micro-macro interactions can modeled as learning interactions (namely the follower/leader dynamics of [31] representing technological imitation of countries belonging to the followers subsystems with respect the leaders one. In this situation the leader subsystem is assumed to influence the net of the followers without being influenced in turn. An important factor affecting the output of the model is that technological imitation is cheaper than innovation, and it may determine higher growth rates in the followers' countries, without forgetting that the growth rates also depend on the social capabilities of the country [1].

The patterns we expect to emerge are either absolute economic convergence, where all the economies under examination approach the mean GDP of the subsystem of leader countries, or club-convergence, where there are few different steady states of GDPs to which subgroups of economies converge in the long run [60, 61].

The KTAP approach helps relating microeconomic and macroeconomic data in the context of a mathematical formulation,. In [31] microeconomic and GDP data for each country are combined; a potential step forward in our understanding of convergence processes adopting the framework we propose, is that of introducing a hierarchical structure in the model, with a lower level for each economy where individual entities (firms and/or economic institutions) enter the picture. This modeling scenario is more complex with respect to the previous one, and it considers heterogeneity among economies as characterized by this lower level, hence allowing micro-founded different growth rates for each country. In this perspective, the KTAP modeling framework is the ideal framework to link microeconomic models of cross-sectional interactions with macroeconomic theories of growth.

A final example of the potential role of KTAP for application to economics is the analysis of systemic banking crisis. In his letter to the G-20, Dominique Strauss-Kahn, Managing Director of the IMF (International Monetary Fund), asks for "developing a reliable early warning and response system [130], able to signal the policy maker the potential arrival of systemic and therefore costly banking crises". In the aftermath of the 2008, this literature has been growing (for a short review see [62]). However, none of the predictive models allow for the possibility that banks, acting as a network, interact with each other and, in turn, this interaction shapes the underlying and unknown distribution of systemic risk endogenously. This, despite the bank industry is known to be clustered according to the bank size, and according to whether banks adopt tools such as securitization and interbank debt [106].

Interactions between social sciences and mechanics: An interesting problem is the interaction between the dynamics from a "soft science", for instance social science, and a mechanical system described by physics, namely a "hard science". This dynamics occurs in human crowds [34, 37] and swarms [40]. A similar problem appears in biology, where biological interactions lead to cell movements, considering that biological dynamics means also communications and learning between cells [112]. Our paper has proposed, only tangentially, tools to tackle this problem. Hence, we simply brought this topic to the reader's attention looking forward to possible activity on this topic. It is worth stressing that this type of modeling needs a detailed analysis of the different types of scaling corresponding to two systems with very different features. Indeed, this appear an open problem worth of further studies. A related problem consists in deriving models at the macroscopic scale from the underlying description at the microscopic scale. Recent developments of the celebrated sixth Hilbert problem [56] can be further developed to tackle this specific problem which involves soft and hard variables.

Validation of models: This topic has been treated in [9], where it has been shown that validation should be based both on the ability of models to reproduce empirical data whenever available as well as to depict emerging behaviors that are qualitatively reproduced for similar causes, but with quantitative differences from one event to the other. Particularly important, once more referring to the framework proposed in [131], is the ability to predict emerging behaviors. This search gives the contribution to experts in social sciences and in economics of explorative tools that can possibly enrich the interaction between hard and soft sciences. Bearing all above in mind, we wish to conclude our paper by the final statement that *human behaviors* should always be taken into account in the development of economic theories. Indeed, this is the strong motivation of the contents of our paper.

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